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Further Applications of Complex Variable Methods to Electrochemical Machining Problems

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SUMMARY

In this paper a complex variable method of solution to problems in electrochemical machining, developed in an earlier paper, is extended to cover a wider class of two-dimensional tool shapes. In particular the case of a straight sided tool with a finite land width is considered.

1. Introduction

Recently a number of papers have appeared in which mathematical techniques are applied to problems in electrochemical machining. In this process, metal is eroded by electrolysis from a work piece which acts as an anode. The cathode is a shaped machine tool which is fed towards the work piece. The electrolyte is pumped through the gap between the electrodes in order to remove the products of erosion. Fitz-Gerald and McGeough [1] have considered the problem of smoothing small surface irregularities on an otherwise plane work surface. Together with Marsh [2], they have also considered the problem of predicting the distribution of irregularities on an initially flat work piece for a given tool profile. In both papers the mathematical analysis was based on the assumption that deviations from a plane face on each electrode are small compared with the electrode gap. By using complex variable methods, Collett, Hewson-Browne and Windle [3] were able to predict the shape of the work piece for two essentially non-linear tool profiles.

In this present paper, the methods employed previously [3] are extended to cover a wider class of two-dimensional tools.

2. The General Configuration and Theory

The two-dimensional configuration to be considered is shown in Fig. 1, where the tool face ABC extends to infinity in two perpendicular directions. The portion BC_- is insulated whilst the portion AB acts as a cathode for the electrochemical process. The tool is fed towards the surface C_+D of the work piece and it is assumed that a steady state has been reached in which this surface moves in the direction of the tool feed at a constant rate equal to the tool feed rate. Taking axes Oxy moving with the tool, as shown in Fig. 1, the configuration is independent of time and the electric field $\boldsymbol{E}=(E_x, E_y, 0)$ satisfies

$$\operatorname{div} \boldsymbol{E}=0, \quad \operatorname{curl} \boldsymbol{E}=0 .$$

These may be satisfied by taking

$$E_x - iE_y = \frac{dw(z)}{dz}, \tag{1}$$

where $z=x+iy$ and $w=\phi+i\psi$. On the electrodes the electric potential ϕ is a constant and we take

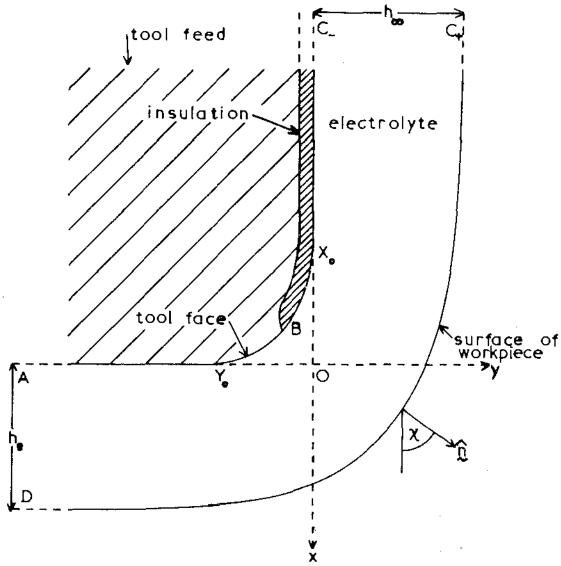

Figure 1. A sketch of the general configuration.

$$\phi = \begin{cases} 0 & \text{on } AB\,, \quad (2) \\ \phi_0 \quad (>0) & \text{on } C_+D\,. \quad (3) \end{cases}$$

On the insulated surface of the tool ψ is a constant and we take

$$\psi = 0 \quad \text{on} \quad BC_-\,. \tag{4}$$

In the steady state, the normal erosion rate, which is proportional to the electric current, is equal to $u \cos \chi$, where u is the tool velocity and χ is the angle between the normal $\hat{n}$ to the work surface and the x-axis. Thus

$$\frac{\partial \phi}{\partial n} = c \cos \chi \qquad \text{on} \qquad C_+D\,, \tag{5}$$

where c is a constant and $\partial/\partial n$ denotes differentiation along the normal. Since

$$\cos \chi = \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial s}\,,$$

where s is the arc length measured along DC_+, (5) may be integrated to give

$$\psi = c(y - h_\infty) \quad \text{on} \quad C_+D\,. \tag{6}$$

Here h_∞ denotes the total overcut C_-C_+. It will also be found convenient to take

$$\psi = -\psi_0(\psi_0 \geqq 0) \quad \text{at} \quad Y_0\,. \tag{7}$$

As $y \to -\infty$, the effects from the portion Y_0C_- are negligible and

$$w \sim c(z - ih_\infty)\,. \tag{8}$$

Thus, using (3),

$$h_e = \phi_0/c\,, \tag{9}$$

where h_e denotes the equilibrium machine gap AD.

Under the transformation

$$\tau = 1 - (1 - k^2) \cos^2\left(\frac{\pi w}{2\phi_0}\right), \tag{10}$$

where

$$0 \leqq k = \tanh\left(\frac{\pi\psi_0}{2\phi_0}\right) \leqq 1\,, \tag{11}$$

the w-plane is mapped into the lower half of the τ-plane as shown in Fig. 2. On Y_0B, $w = i\psi$ $(-\psi_0 \leqq \psi \leqq 0)$ and (10) gives

$$\psi = \frac{2\phi_0}{\pi}\,\xi(\tau)\,, \tag{12}$$

where

$$\xi(\tau) = \log\left[\frac{(1-\tau)^{\frac{1}{2}} - (k^2-\tau)^{\frac{1}{2}}}{(1-k^2)^{\frac{1}{2}}}\right]. \tag{13}$$

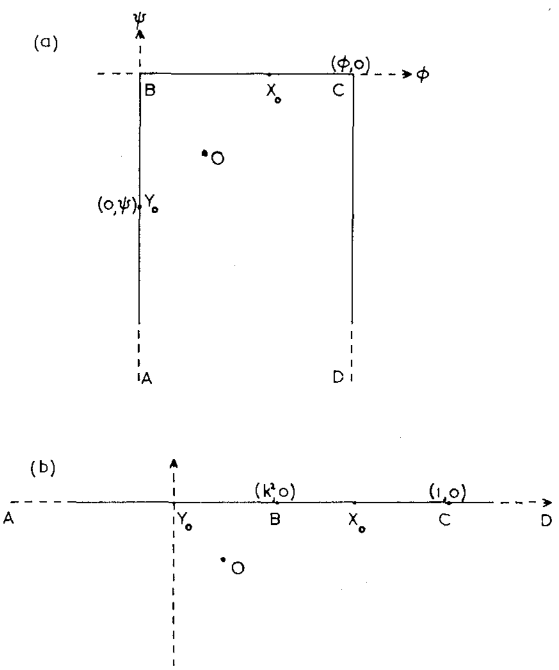

Figure 2. (a) The w-plane. (b) The τ-plane.

Similarly on BC, where $w = \phi\,(0 \leqq \phi \leqq \phi_0)$,

$$\phi = \frac{2\phi_0}{\pi}\,\eta(\tau)\,, \tag{14}$$

where

$$\eta(\tau) = \sin^{-1}\left[\left(\frac{\tau-k^2}{1-k^2}\right)^{\frac{1}{2}}\right] \quad \left(0 \leqq \eta \leqq \frac{\pi}{2}\right). \tag{15}$$

Putting

$$z = \frac{w}{c} + ih_\infty + f(\tau)\,, \tag{16}$$

(8) implies that (17)

$$f(\tau) \to 0 \quad \text{as} \quad |\tau| \to \infty\,.$$

We also require

$$f(\tau) \text{ bounded for } \tau \neq 1 . \tag{18}$$

Then on AY_0, $x=0$, $\phi=0$ and so

$$\operatorname{Re}\{f(\tau)\}=0 . \tag{19}$$

We may therefore continue $f(\tau)$ across AY_0 by reflection. Since $\bar{\tau}\,e^{-2\pi i}$ is the reflection of τ in AY_0,

$$f(\bar{\tau}e^{-2\pi i}) = -\overline{f(\tau)} ,$$

and in particular

$$\bar{f}(\tau) = -f(\tau e^{-2\pi i}) \quad \text{on} \quad Y_0 D . \tag{20}$$

On $X_0 C$, $y=0$, $\psi=0$ and so

$$\operatorname{Im}\ \{f(\tau)\} = -h_\infty$$

Using (20) this gives

$$f(\tau)+f(\tau e^{-2\pi i}) = -2ih_\infty \quad \text{on} \quad X_0 C . \tag{21}$$

On CD, $\psi=c(y-h_\infty)$ and so

$$\operatorname{Im}\{f(\tau)\}=0 .$$

Using (20) this gives

$$f(\tau)+f(\tau e^{-2\pi i})=0 \quad \text{on} \quad CD . \tag{22}$$

In order to proceed further we consider the case in which the portion $Y_0\ X_0$ of the tool face consists of $(n+1)$ straight edges $G_r G_{r+1}$ $(0 \leqq r \leqq n)$ as shown in Fig. 3, where the vertices G_0, G_m and G_{n+1} are to be identified with the points Y_0, B and X_0 respectively.

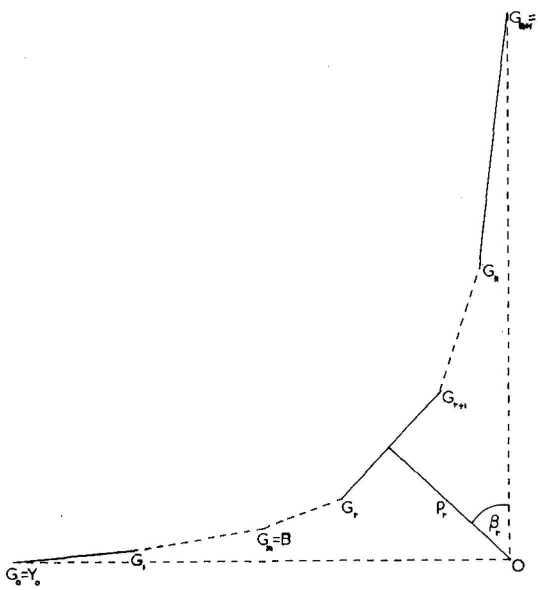

Figure 3. The straight-sided configuration for the portion $Y_0 X_0$ of the tool face.

The side $G_r G_{r+1}$ is specified by the length p_r of the perpendicular from 0 onto that side and by the angle β_r between the perpendicular and the negative x-axis. We suppose that the points G_r map into the points $\tau=\tau_r$ (real) where

$$\tau_0 = 0 \quad \text{and} \quad \tau_m = k^2 . \tag{23}$$

Then on $G_r G_{r+1}$

$$p_r + x \cos\beta_r + y \sin\beta_r = 0 ,$$

and so

$$\operatorname{Re}\{e^{-i\beta_r} z\} = -p_r .$$

Using (16) this becomes

$$\operatorname{Re}\{e^{-i\beta_r} f(\tau)\} = g_r(\tau), \qquad \tau_r < \tau < \tau_{r+1} , \tag{24}$$

where

$$g_r(\tau) = -p_r - \frac{\phi}{c}\cos\beta_r - \left(\frac{\psi}{c} + h_\infty\right)\sin\beta_r \qquad (0 \leqq r \leqq n) . \tag{25}$$

Thus for $0 \leqq r \leqq m-1$ (i.e. on $Y_0 B$ where $\phi = 0$ and $\psi = 2\phi_0 \xi(\tau)/\pi$),

$$g_r(\tau) = -p_r - \left(\frac{2\phi_0}{\pi c}\xi(\tau) + h_\infty\right)\sin\beta_r , \tag{26}$$

and for $m \leqq r \leqq n$ (i.e. on BX_0 where $\phi = 2\phi_0 \eta(\tau)/\pi$ and $\psi = 0$),

$$g_r(\tau) = -p_r - \frac{2\phi_0}{\pi c}\eta(\tau)\cos\beta_r - h_\infty \sin\beta_r . \tag{27}$$

Using (20), (24) gives

$$f(\tau) - e^{2i\beta_r} f(\tau e^{-2\pi i}) = 2 e^{i\beta_r} g_r(\tau) , \qquad \tau_r < \tau < \tau_{r+1} \quad (0 \leqq r \leqq n) . \tag{28}$$

Equations (21), (22) and (28) are to be solved subject to (17) and (18). This is a non-homogeneous Hilbert problem [4] whose solution depends on the function

$$f_0(\tau) = \prod_{r=0}^{n+1} (\tau - \tau_r)^{(\alpha_r/\pi) - 1} \tag{29}$$

where the α_r (assumed either greater or equal to π) are the external angles subtended by the tool face at the vertices G_r. In terms of the angles β_r, we have

$$\begin{aligned} \alpha_0 &= \pi + \beta_0 , \\ \alpha_r &= \pi + \beta_r - \beta_{r-1} , \qquad 1 \leqq r \leqq n , \\ \alpha_{n+1} &= 3\pi/2 - \beta_n . \end{aligned} \tag{30}$$

The function $f_0(\tau)$ is in fact a solution of (21), (22), (28) after their right hand sides have been replaced by zero. The solution for $f(\tau)$ is then

$$f(\tau) = \frac{f_0(\tau)}{2\pi i}\left\{ \int_1^{\tau_{n+1}} \frac{-2ih_\infty}{f_0(t)(t-\tau)}\, dt + \sum_{r=0}^{n} \int_{\tau_{r+1}}^{\tau_r} \frac{2 e^{i\beta_r} g_r(t)}{f_0(t)(t-\tau)}\, dt \right\} . \tag{31}$$

The method will be illustrated in the next section.

3. The Case of a Straight Sided Tool with Finite Land Width

The tool is shown in Fig. 4. The insulation extends to within a distance ϖ (the land width) of the corner Y_0. Here

$$m = 1 , \quad n = 0 , \quad \alpha_0 = \frac{3\pi}{2} , \quad \alpha_1 = \pi , \quad p_0 = 0 , \quad \beta_0 = \pi/2 . \tag{32}$$

In this case, (29) gives

$$f_0(\tau) = \tau^{\frac{1}{2}}, \tag{33}$$

and (31) becomes

$$f(\tau) = \frac{\tau^{\frac{1}{2}}}{2\pi i}\left\{\int_1^{k^2} \frac{-2ih_\infty}{t^{\frac{1}{2}}(t-\tau)}\,dt + \int_{k^2}^0 \frac{2e^{i\pi/2}g_0(t)}{t^{\frac{1}{2}}(t-\tau)}\,dt\right\}, \tag{34}$$

where, from (26),

$$g_0(t) = -\frac{2\phi_0}{\pi c}\,\xi(t) - h_\infty\,. \tag{35}$$

After some simplification (34) and (35) give

$$f(\tau) = -\frac{h_\infty}{\pi}\log\left(\frac{\tau^{\frac{1}{2}}+1}{\tau^{\frac{1}{2}}-1}\right) + \frac{\phi_0}{\pi^2 c}\int_0^{k^2}\log\left(\frac{\tau^{\frac{1}{2}}+t^{\frac{1}{2}}}{\tau^{\frac{1}{2}}-t^{\frac{1}{2}}}\right)\frac{dt}{(1-t)^{\frac{1}{2}}(k^2-t)^{\frac{1}{2}}}, \tag{36}$$

where the branches are chosen so that $f(\tau)\to 0$ as $|\tau|\to\infty$.

As $\tau\to 0$, it may be shown that

$$f(\tau) \sim i\left(\frac{\psi_0}{c} - h_\infty\right) + \frac{2}{\pi}\left(\frac{2\phi_0 K}{\pi c} - h_\infty\right)\tau^{\frac{1}{2}} - \frac{i\phi_0\tau}{\pi c k} + O(\tau^{\frac{3}{2}}), \tag{37}$$

where

$$K = \int_0^{\pi/2} \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{1}{2}}}\,. \tag{38}$$

is the complete elliptic integral of the first kind. From (10) we see that

$$w \sim -i\psi_0 + \frac{i\phi_0\tau}{\pi k} + O(\tau^2) \quad \text{as} \quad \tau\to 0\,. \tag{39}$$

Using (37) and (39), (16) gives

$$z \sim \frac{2}{\pi}\left(\frac{2\phi_0 K}{\pi c} - h_\infty\right)\tau^{\frac{1}{2}} + O(\tau^{\frac{3}{2}}) \quad \text{as} \quad \tau\to 0\,. \tag{40}$$

However it is evident that the mapping from the z-plane to the τ-plane ceases to be conformal at Y_0 in such a way that

$$z \propto \tau^{\frac{3}{2}} \quad \text{as} \quad \tau\to 0\,. \tag{41}$$

Comparing (40) with (41) we see that

$$h_\infty = \frac{2\phi_0 K}{\pi c}\,. \tag{42}$$

Using (9), this may be expressed in the alternative form

$$\frac{\text{total overcut}}{\text{machining gap}} = \frac{h_\infty}{h_e} = \frac{2K}{\pi}\,. \tag{43}$$

At B, $z=-\varpi$, $w=0$ and $\tau=k^2$. Thus (16) gives

$$\varpi = -ih_\infty - f(k^2)\,,$$

and using (36) together with (9) and (42), we obtain

$$\frac{\text{landwidth}}{\text{machine gap}} = \frac{\varpi}{h_e} = \frac{2}{\pi^2}\left[K \log\left(\frac{1+k}{1-k}\right) - k\int_0^1 \log\left(\frac{1+q}{1-q}\right)\frac{q\,dq}{(1-k^2q^2)^{\frac{1}{2}}(1-q^2)^{\frac{1}{2}}}\right] \tag{44}$$

On the work piece surface C_+D, (6) and (16) give

$$\left.\begin{aligned} x &= \phi_0/c + f(\tau)\,, \\ y &= h_\infty + \psi/c\,. \end{aligned}\right\} \tag{45}$$

Thus, using (9), (10), (36) and (42), we obtain parametric equations of the work surface in the form

$$\frac{x}{h_e} = 1 - \frac{2K}{\pi^2}\log\left(\frac{\tau^{\frac{1}{2}}+1}{\tau^{\frac{1}{2}}-1}\right) + \frac{1}{\pi^2}\int_0^{k^2}\log\left(\frac{\tau^{\frac{1}{2}}+t^{\frac{1}{2}}}{\tau^{\frac{1}{2}}-t^{\frac{1}{2}}}\right)\frac{dt}{(1-t)^{\frac{1}{2}}(k^2-t)^{\frac{1}{2}}}\,, \tag{46}$$

$$\frac{y}{h_e} = \frac{2}{\pi}\left\{K - \log\left[\frac{(\tau-1)^{\frac{1}{2}} + (\tau-k^2)^{\frac{1}{2}}}{(1-k^2)^{\frac{1}{2}}}\right]\right\}, \tag{47}$$

where $\tau\ (>1)$ is real and the parameter k is related to the land width ϖ by equation (44). The configuration for the case $k=0.99$ is shown in Fig. 4, where $\varpi = 1.857\ h_e$ and $h_\infty = 3.357\ h_e$.

The variation of the total overcut h_∞ with the land with the land width is shown in Fig. 5(a). It may be shown from (43) and (44) that

$$\frac{h_\infty}{h_e} \sim 1 + \tfrac{1}{4}\left(\frac{3\pi\varpi}{h_e}\right)^{\frac{2}{3}} + O\left(\left[\frac{\varpi}{h_e}\right]^{\frac{4}{3}}\right) \quad \text{for } \varpi \ll h_e \quad (k \to 0)\,, \tag{48}$$

and

$$\frac{h_\infty}{h_e} \sim \left(\frac{2\varpi}{h_e}\right)^{\frac{1}{2}} + \frac{2}{\pi}\log 2 + O\left(\left[\frac{h_e}{\varpi}\right]^{\frac{1}{2}}\right) \quad \text{for } \varpi \gg h_e \quad (k \to 1)\,. \tag{49}$$

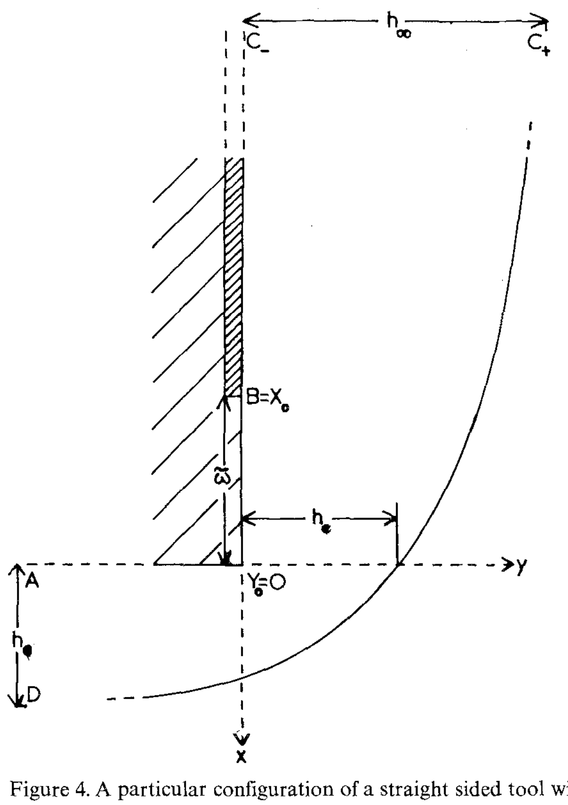

Figure 4. A particular configuration of a straight sided tool with insulation.

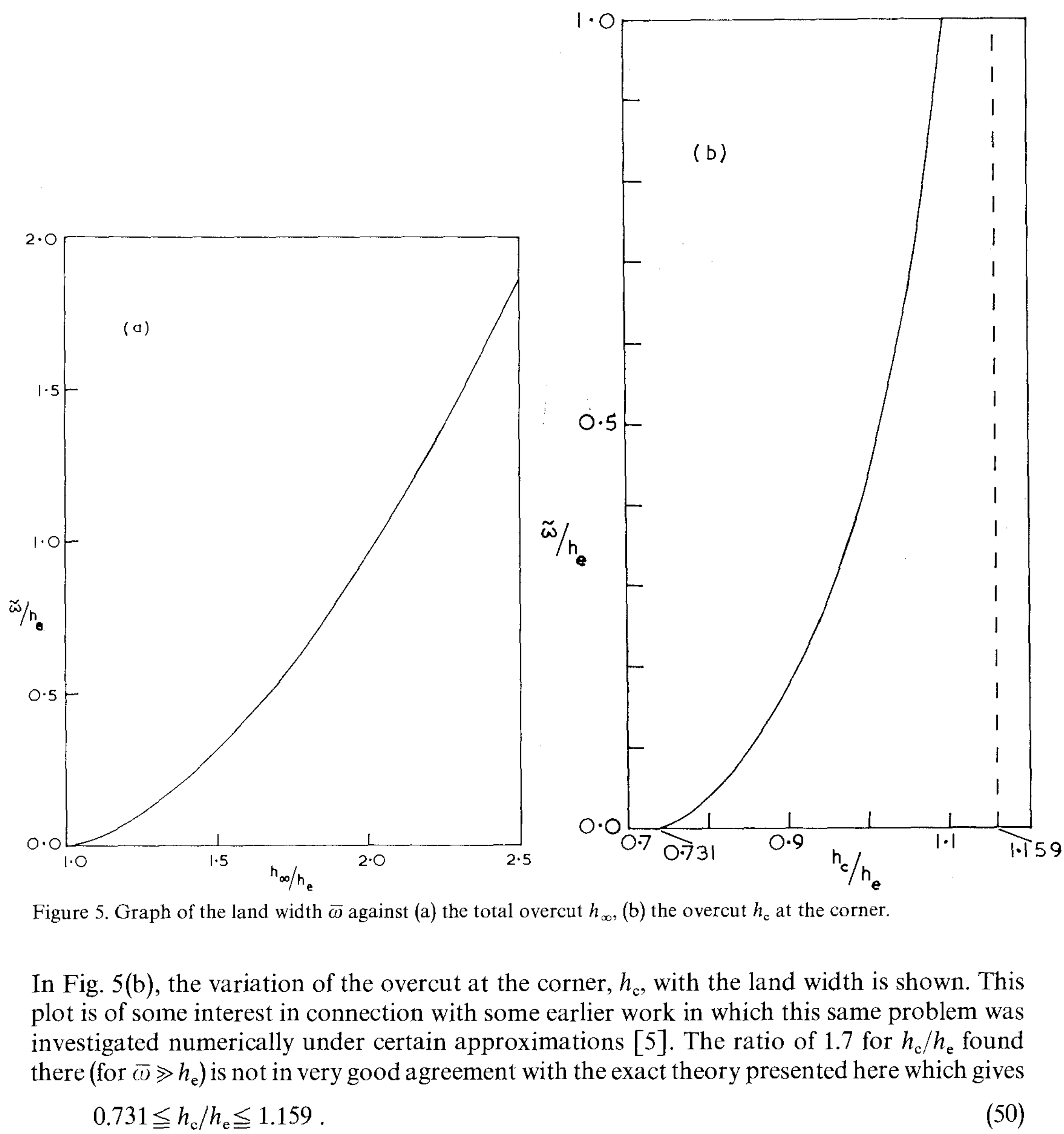

Figure 5. Graph of the land width $\bar{\omega}$ against (a) the total overcut h_∞, (b) the overcut h_c at the corner.

In Fig. 5(b), the variation of the overcut at the corner, h_c, with the land width is shown. This plot is of some interest in connection with some earlier work in which this same problem was investigated numerically under certain approximations [5]. The ratio of 1.7 for h_c/h_e found there (for $\bar{\omega} \gg h_e$) is not in very good agreement with the exact theory presented here which gives

$$0.731 \leqq h_c/h_e \leqq 1.159 . \tag{50}$$

The two extreme values in (50) correspond to the cases $\bar{\omega} = 0$ and $\bar{\omega} = \infty$ respectively. An alternative treatment for these two cases has been given earlier [3].

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